ON THE CLASS OF GRAPHS ZP

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SUPERVISORY AND EXAMINING COMMITTEE

Brendan Kyle Andrusiak, candidate for the degree of **Master of Science in Mathematics,** has presented a thesis titled, ***On the class of graphs ZP,*** in an oral examination held on **April 26, 2024**. The following committee members have found the thesis acceptable in form and content, and that the candidate demonstrated satisfactory knowledge of the subject material.

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Abstract

The following research was primarily focused on the class of graphs denoted by ZP. Let $G = (V, E)$ be a graph made up of vertices V and edges E. The path cover number and zero forcing number are two graph parameters that have been of recent research interests and are closely related. Zero forcing at it's most rudimentary, is a graph colouring game. There is a significant preexisting body of work on zero forcing, which includes relations between zero forcing and path cover numbers (denoted by $Z(G)$ and $P(G)$, respectively), as well as a relation between zero forcing and a notion of maximum nullity of a graph. One natural question along these lines that emerged was to impose equality conditions between $Z(G)$ and $P(G)$, and assuming these equality constraints hold for both G and all induced subgraphs of G, what class of graphs might arise and what is special about said graphs? Thus, we study the class ZP in which the zero forcing number and the path cover number are equal over all induced subgraphs. As many graphs are known to belong to ZP, such a trees, cycles, and cacti, these graphs are an excellent starting point for study. Hence, the cycle graph, denoted by C_n, provide the primary point of study early on in the research process. As C_n is a graph known to belong to ZP, we add interior chords to the cycle graph in many different orientations and in many numbers, then examine the resulting changes in both $Z(G)$ and $P(G)$. We then consider analyzing graphs that belong to ZP by conditioning on possible values of the path cover number, namely assuming $P(G) = 2$ and $P(G) = 3$. Finally, graph operations and their effect on graphs in ZP are considered. Of particular importance are the vertex and edge-sum operations. Ultimately, we are able to prove that the vertex or edge-sums of graphs in ZP do indeed remain in the class ZP.

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Contents

List of Figures

Chapter 1

Introduction

When we consider graphs in this thesis, we speak of so-called simple graphs. That is, we define a graph G as an ordered pair $G = (V, E)$, with V a nonempty and finite set of vertices and E a set of edges consisting of a subset of unordered pairs of vertices, without self loops nor multiple edges. Most graphs in this thesis will be assumed to be connected (namely there is a path between any pair of vertices), unless otherwise stated. Finite simple graphs arise in many fields in mathematics. Typically, they appear as a model for a particular application (such as: electrical or computer networks, geometry and discrete optimization). In linear algebra graphs appear in numerous situations, particularly when such studies involve patterns of the non-zero (or signs) of the entries in a real symmetric matrix.

Given a real symmetric $n \times n$ matrix A whose $(i, j)^{th}$ entry is denoted by a_{ij}, for $i, j = 1, 2, \ldots, n$ (we set $A = [a_{ij}]$ for short), we define the graph of A, written

$G(A)$, to have vertex set $V = \{1, 2, \ldots, n\}$ and for $i \neq j$, $\{i, j\} \in E$ if and only if $a_{ij} \neq 0$. A natural line of inquiry is to consider properties of all symmetric matrices B with $G(B)$ isomorphic to $G(A)$ (written as $G(B) \cong G(A)$ for short). We call this class of matrices $S(G)$ with $G \cong G(A)$. It has been known for some time that both the path cover number (denoted by $P(G)$ and defined below) and the zero forcing number (denoted as $Z(G)$ and defined below) of a graph G are intimately tied to certain spectral information associated with the matrices in $S(G)$, for example both have a relationship with the maximum possible nullity of a graph (see [1, 17]).

Our interest lies in studying the fascinating relationship between $P(G)$ and $Z(G)$. For any graph G, the zero forcing number is always an upper bound for the path cover number [1] and studying the extremal graphs for which these numbers are equal is an important and still rather unresolved line of inquiry. To this end, our focus is investigating the collection of all graphs G such that $P(H) = Z(H)$ for all induced subgraphs H of G. This class of graphs, denoted by ZP, represents a strict subset of the extremal graphs for which $Z(G) = P(G)$ (see an example in Section 2.4 to illustrate this claim). The class ZP was defined in [14] where it was shown that all cacti belong to the class ZP. In fact, earlier Row [20] noted that for any cactus graph G we have $Z(G) = P(G)$. In some sense these results were the springboard for the current research conducted in this thesis.

1.1 Thesis Organization

It is known that trees are in ZP, and a brief examination confirms this fact, as both the path cover and zero forcing number are both equal to one. It is natural then, that after examining a class of graphs with both parameters equal to one, that the next graphs to examine would be those with zero forcing and path cover number equal to two. Once again, Row made a number of interesting contributions on this topic. In [21] Row introduces the notion of z-induced path covers (more details are provided in Chapter 2), that is, a minimum path cover follows the same lists of vertices associated with a the paths induced by the forces from a minimum zero forcing set. This fact makes computing path covers much easier, and thus graphs of two parallel paths became a focal point, as they were known to have a zero forcing number of two. Thus, in Chapter 2 we review more results from Row [21], as well as relevant history and setup and introduce the concept of terminality.

In Chapter 3, we examine the class of graphs known as cycles very closely. It is known that cycles are in ZP. We use the cycle as a seed graph where we then establish results concerning cycles with various configurations of interior chords and consider how our central graph parameters change when graphs are planar or maximally outerplanar. Now, during this examination, we also study graphs with path cover number equal to two, which was previously noted as the next logical step in research. We introduce novel results concerning cycles with interior chords and graphs that are maximally outerplanar. As previously mentioned in the beginning, we attempt completely classify the class ZP, but no general resolution is given. As this may very well include a very large amount of different graphs, it is natural to restrict certain graph parameters to narrow down the potential options. After the path cover number equal to one case, we focus on the case with path cover number equal to two, which we explore in Section 3.3. After reviewing information from [19] and [11] as well as a novel result, we then look at the pinwheel graph and separator triangles, which are instrumental in examining graphs with path cover equal to three.

In Chapter 4, we consider the notion of graph operations. Multiple graph operations are reviewed, as is their relation to the class ZP. This chapter is where the "main" result is presented. We are able to establish that the vertex-sum and edge-sum of two graphs in ZP remain in ZP.

Chapter 2

Preliminary Analysis

2.1 History and General Setup

In this section and for the remainder of this thesis, we will discuss certain graph parameters and their relation to each other. The most notable parameters that will be discussed are the *zero forcing number* and *path cover number.* The zero forcing number and a couple other relevant terms are defined in Section 2 of paper [1] by Barioli et al., and are as follows:

Definition 2.1.1 *Color-change rule: If G is a graph with each vertex coloured either blue or white, and u is a blue vertex of G and exactly one vertex v adjacent to u is white, then the vertex v is coloured blue.*

Definition 2.1.2 *Suppose a subset B of V is initially coloured blue. The derived coloring of B is the resulting set of blue vertices after performing all possible colour* *changes until no more changes are possible.*

Definition 2.1.3 *A zero forcing set for a graph G is a subset of the vertices Z, such that if all the vertices in Z are initially coloured blue, then the derived colouring of Z is V.*

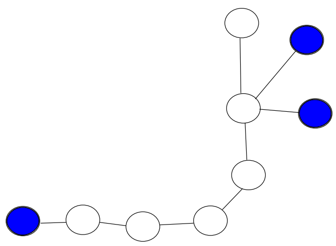

Figure 2.1: A zero forcing set for a tree with $Z(G) = 3$

With these definitions in mind, we set the notation $Z(G)$ to equal the minimum cardinality over all zero forcing sets $Z \subseteq V(G)$, and call $Z(G)$ the *zero forcing number* of a graph G.

Now, the second graph parameter mentioned above is the path cover number. We use the definition of Barioli et al. [5].

Definition 2.1.4 *The path cover number of a graph G is the minimum number of vertex-disjoint paths occurring as induced subgraphs of G which cover all the vertices* *of* G.

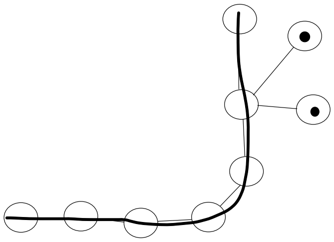

Figure 2.2: Path Cover for a tree with $P(G) = 3$

Further known definitions will be defined as they become relevant to this thesis. However, $Z(G)$ and $P(G)$ can be considered as the central parameters from which the material in this thesis will be derived. Initially, the concept of zero forcing was introduced as a topic for further research. Many relations between zero forcing and other graph parameters are already known. For example, the relation between nullity and zero forcing for certain graphs had been established by Barioli et al. in [1]. In our initial period of research on zero forcing, applications into the fields of physics (see [7] and computer science (see [8]) are already known and are very interesting. Naturally, we seek to establish novel relationships between zero forcing and other graph parameters or find applications for zero forcing in other areas of science. While pursuing the former, the class ZP was suggested as an area to research further. Being a special subset of graphs, it follows easily from the definition that if G is a member of the class ZP, then the relation $Z(G) = P(G)$ must always hold. Later, we re-establish that trees are indeed in the class ZP.

The central object of study in this thesis is class of graphs belonging to ZP. If $G = (V, E)$ is a graph and $V' \subset V$, then the *subgraph of* G *induced by* V' is the subgraph with vertex set V' and any edges from E with both endpoints in V'. In general, a subgraph H is called an induced subgraph of G if H is induced by some subset of the vertices of G. We now define this important class of core study in this thesis.

Definition 2.1.5 *A graph* G *is said to belong the class* ZP *if* $Z(H) = P(H)$ *for all induced subgraphs* H *of* G.

In the next section, we discuss relevant definitions, facts, and other results corresponding to both the zero forcing number and path cover number.

2.2 Consequences, facts and other results

As previously mentioned, Row has made many contributions on this topic of research. As we continue to accrue more known facts and results, we also start to extend some of these known results. This can be seen in Proposition 2.2.2.

The following result by Row [21] is fundamental to our research

Theorem 2.2.1 *[21] If G is a graph with the property that every path cover is z-induced (that is, every path cover corresponds to the forcing chains of a minimum zero forcing set), then $Z(G) = P(G)$.*

In fact, we can prove a slightly stronger statement along these lines.

Proposition 2.2.2 *Suppose G is a graph. Then $Z(G) = P(G)$ if and only if G contains an optimal path covering (number of paths equals $P(G)$) that is z-induced.*

Proof. We already know that $Z(G) \geq P(G)$ for every graph G. Suppose G contains an optimal path covering (number of paths equals $P(G)$) that is z-induced. Then there is a zero forcing set X of size $P(G)$, from which it follows that $Z(G) = P(G)$. On the other hand, suppose $Z(G) = P(G)$. Then, the forcing chains from the zero forcing set of size $Z(G)$ produces an optimal (or minimum) path covering of G that is z-induced. ∎

Proposition 2.2.2 serves as the starting point of our investigation. Beyond this, Proposition 2.2.2 is the first effort to combine multiple previously known results to determine if any such fusion of results was possible, and if so, to what end.

2.3 Notions of Terminality

As previously stated, multiple results in this area of research are examined and interpreted. As we have also centre our research around two key graph parameters on different subsets of graphs, there are several known results that became important later in this research process. In this section, we will concentrate on a more in-depth relation between the zero forcing and path cover numbers of a graph.

We now recall the notions of "terminal" within the contexts of zero forcing chains and path coverings (see also [19] and [5]).

We discuss the concept of terminal with respect to optimal path coverings in a given graph G.

Definition 2.3.1 *Given a graph* $G = (V, E)$ *and* $v \in V$*, we call* v p-terminal *(p for path) if* v *is the endpoint of some path in a minimum path cover in* G*. Additionally, a p-terminal vertex* v *is called:*

- doubly p-terminal *if* v *is p-terminal and* $\{v\}$ *is a path in some minimum path cover in* G*;*
- simply p-terminal *if* v *is p-terminal and* v *is not doubly p-terminal.*

We now turn our attention to the topic of terminality with respect to optimal zero forcing sets in a given graph G.

Definition 2.3.2 *Given a graph* $G = (V, E)$ *and* $v \in V$*, we call* v z-terminal *(z for zero forcing) if* v *is the endpoint of some forcing chain associated with a minimum zero forcing set in* G*. Additionally, a z-terminal vertex* v *is called:*

- doubly z-terminal *if v is z-terminal and v never performs a force in G for some minimum zero forcing set;*

- simply z-terminal *if v is z-terminal and v is not doubly z-terminal.*

Note that any z-terminal vertex must be a member of an optimal zero forcing set. We know that for any graph G we have $P(G) \leq Z(G)$ and that equality need not hold in general. Along the same lines, the concepts of p-terminal and z-terminal need not be equivalent. For example, suppose $G = K_{2,3}$, and v is any vertex of degree two. Then, we know that $2 = P(G) < Z(G) = 3$ and it is easy to confirm that v is simply p-terminal, but v is doubly z-terminal. Similarly, suppose G is the 3-by-3 grid or better known as the Cartesian product of P_3 with itself (namely, $P_3 \Box P_3$). Then it is well-known that $2 = P(G) < Z(G) = 3$. Assume v is the unique vertex of degree four in G. Then v is p-terminal (in fact simply p-terminal) and is also z-terminal. These relationships and various forms of $z-$ and $p-$terminality are important as it leads to the next important result.

Theorem 2.3.3 *Suppose G is a graph in ZP. Then the notions of p-terminal and z-terminal are equivalent.*

Proof. Since $P(G) = Z(G)$, it follows from Proposition 2.2.2 that G contains an optimal path covering that is z-induced. Hence if a vertex v is z-terminal then it is also p-terminal. Furthermore it also follows easily that if v is doubly z-terminal, then it is doubly p-terminal as well.

It follows that $P(G) = P(G - v) + 1$ if and only if v is doubly p-terminal in some minimum path cover for G. Since G is in ZP it follows $Z(G) = Z(G - v) + 1$, so v is also doubly z-terminal.

On the other hand, suppose v is simply p-terminal. Then it follows that $P(G) - P(G - v) \leq 0$. Suppose $P(G) - P(G - v) = -1$, then it is known that v is not p-terminal, which contradicts our assumption. So we must have $P(G) = P(G - v)$. Since G is in ZP we also have $Z(G) = Z(G - v)$. Now let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ be a minimum path over for G with v as an endpoint of P_1. Since v is not doubly terminal, we know that P_1 contains at least two vertices. Let w be the other distinct endpoint of P_1. Observe that since G is in ZP, it follows that v is part of a zero forcing set for the graph $G - w$ induced by the minimum path covering $\mathcal{P}$, but it may not be a minimum zero forcing set in the case when w is doubly z-terminal. Since there are no edges between the vertices of P_1 in G (as P_1 is an induced path), two possible cases arise. Since $\mathcal{P}$ induces a path covering for the graphs $G - v$ and $G - w$, either v is a part of a zero forcing set induced by $\mathcal{P}$, which implies v is z-terminal as $P(G) = P(G - v) = Z(G) = Z(G - v)$; or w is part of a zero forcing set induced by $\mathcal{P}$. In the latter case, since the reverse of any such zero forcing set is also a zero forcing set, it follows that v is part of a zero forcing set for G of size k (the number of paths in $\mathcal{P}$). Hence v is z-terminal for G, as desired. ▮

An interesting consequence of the above analysis is an extension of Proposition 2.2.2.

Corollary 2.3.4 *Suppose G is a graph in the class ZP. Then every optimal path covering (number of paths equals $P(G)$) is z-induced.*

As we previously alluded to, this corollary proved vital in determining if various graphs or graph families belonged to ZP. During the examination of said graphs, only an optimal zero forcing set is needed in some sense. From this, all that remained was to confirm that the path cover was z-induced.

2.4 Other Facts and Existing Results

In this section we consider various classes of graphs that belong to the class ZP. As we have already noted, any tree (or connected acyclic) graph is a member of the class ZP. In fact, this claim also follows from Proposition 2.2.2, as has been shown that any minimum path covering of a tree is also a z-induced path covering. We can extend this argument beyond trees by considering a forest, or a disjoint union of trees, since both P and Z are additive graph parameters across connected components. Furthermore, using the 3-by-3 grid graph as noted in Section 2.2, if we form a new graph G by adding pendant vertices to opposite corners of this grid, then it follows that $Z(G) = P(G)$, but G is not in ZP as the 3-by-3 grid itself has a path cover number strictly less than its zero forcing number.

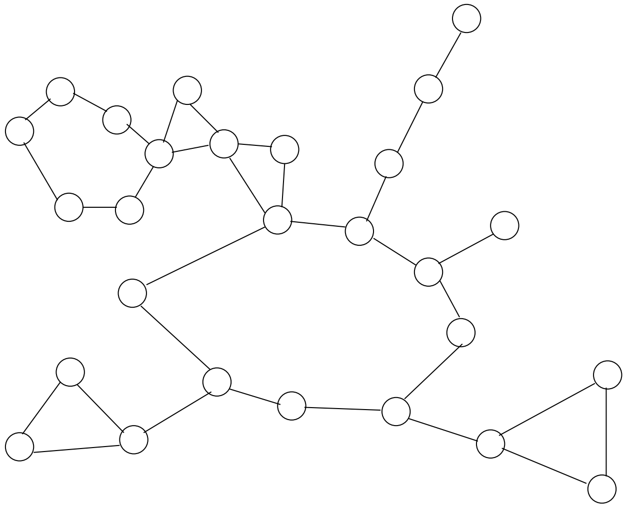

Figure 2.3: Cactus

Building from trees, we have the class of unicyclic graphs. A graph is called unicyclic if it contains exactly one cycle. It is not difficult to verify that every unicyclic graph G can be obtained from a tree by adding a single edge. In the paper [20], it was shown that $Z(G) = P(G)$ for any unicyclic graph G. In fact, a stronger result holds for unicyclic graphs. Suppose G is a unicyclic graph and let H be a proper induced subgraph of G. Then two cases arise: (1) H is unicyclic; or (2) H is a forest. However, in either case it follows from the discussion above that $P(H) = Z(H)$. Hence any unicyclic graph belongs to the class ZP.

The next class of graphs we consider here are commonly known as cacti. A graph

G is called a *cactus* if it is a connected graph in which any two simple cycles have at most one vertex in common. Equivalently, it is a connected graph in which every edge belongs to at most one simple cycle, or in which every block is an edge or a cycle. Regarding cactus graphs, the main result that we focus on here, is found in [20, Theorem 3.3].

Theorem 2.4.1 *[20] Suppose G is a cactus graph. Then $Z(G) = P(G)$.*

As a matter of completeness, we note that to prove Theorem 2.4.1, Row makes use of induction on the number of cycles within the graph. Following similar lines of reasoning as presented above for unicylic graphs, it turns out that all cacti belong to the class ZP. The key ingredient to verifying this is naturally Theorem 2.4.1.

Now, an important fact on this topic can be found in [14], In this work Gentner et al propose a forbidden subgraph for a certain subclass of the class ZP. In this paper, see [14, Proposition 6] it is shown that a graph is a cactus if and only if it is $K_4 \bigcup \Theta(l_1, l_2, l_3)$-free, where $l_1, l_2, l_3 \in \mathbb{N}$ and $l_2, l_3 \geq 3$. As defined in [14], for positive integers l_1, l_2 and l_3 with $l_2, l_3 \geq 2$, let Θ (l_1, l_2, l_3) be the graph that has two vertices of degree 3 that are linked by three paths of length l_1, l_2, and l_3, respectively, whose internal vertices are all of degree 2.

Concerning the class ZP and cacti, another important result is presented in [14, Theorem 7].

Theorem 2.4.2 *If G is a graph such that every cycle of G is induced, then the following statements are equivalent.*

1. *$G \in ZP$,*
2. *G is a cactus,*
3. *G is $K_4 \bigcup \Theta(l_1, l_2, l_3)$-free, where $l_1, l_2, l_3 \in \mathbb{N}$ and $l_2, l_3 \geq 3$.*

These two results were certainly important for our research. In Section 3.2, we examine the relationships between cycles and interior chords, we pay particular attention to $Z(G)$ when we have intersecting interior chords versus chords with no intersection. Thus, if the construction of said graph contained $\{K_4\}$ as a subgraph, we immediately knew that it could not be a cactus, but also would not be in the class ZP. More generally, speaking since belonging to the class ZP is clearly a condition inherited by induced subgraphs, it follows that if a graph H does not belong to ZP, then any graph G containing H as an induced subgraph will not be a member of the class ZP.

Furthermore, one of the main contributions of this thesis approaches a proof of Row's Theorem 2.4.1 differently (see Chapter 4), and in the process we are able to streamline portions of its verification.

Chapter 3

Graphs in ZP with Small Path Cover Number

3.1 History and Background

In this section, we begin by examining some relevant definitions and notions that will support our discussion in the following chapter. As there are many definitions in this section, several diagrams are provided to aid with clarity. Further, in this chapter we examine cycles with interior chords, certain families of sparse graphs (e.g. trees), graphs with path cover number at most two, and the important pinwheel graph. Exploration of the pinwheel graph graph also allows us to examine what was thought to be a potential forbidden subgraph regarding potential membership in the class ZP; the separator triangle.

Definition 3.1.1 *Let* v_1 *and* v_2 *be adjacent vertices in* C_n*. Now, let* u_1 *and* u_2 *be* *two adjacent vertices in* C_n *which are not adjacent to* v_1 *or* v_2*. We say that interior chords intersect if they cross while connecting two previously non-adjacent vertices. In this case, if an interior chord connected* u_1 *and* v_2 *and another connected* u_2 *and* v_1*, then the chords would intersect, unlike, for example, interior chords connecting* u_1, v_1 *and* u_2, v_2 *(see, for example Figure 3.1).*

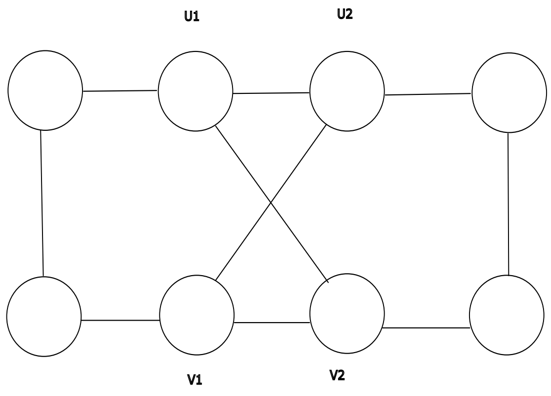

Figure 3.1: Intersecting interior chords

Definition 3.1.2 *We say interior chords are adjacent if they connect adjacent vertices in* C_n*. Consider two pairs of adjacent vertices* u_1, u_2 *and* v_1, v_2*, respectively, in*

C_n such that neither u_1, u_2 are adjacent to v_1, v_2. Connect u_1 and v_1 via an interior chord denoted t and connect likewise u_2 and v_2 in a similar manner with interior chord denoted s. By construction, we say that s and t are adjacent interior chords.

Definition 3.1.3 *We say a graph is planar if it can be embedded in a plane so that edges only intersect at vertices. Further, we say that a graph is outerplanar if it can be drawn in the plane with all vertices adjacent to the infinite (or outer) face. Finally, we say a graph G is* maximally outerplanar *(and write MOP for short) if it is not possible to add another edge to the graph G without violating the property of being outerplanar.*

Often an MOP graph G is referred to as a polygon triangulation, since each bounded face (or closed region in the plane determined by a planar embedding of G) must be a triangle. To this end a triangular face is called a *separator triangle* if it has no edges lying in the infinite face. The number of separator triangles in an MOP G is denoted by $t(G)$ (we may just use t for short when the graph G is easily determined from context).

Definition 3.1.4 *An MOP graph G is called* serpentine *if $t(G) = 0$.*

Following the work in [16], given a graph G, the total zero forcing number, $Z_T(G)$, is the size of the smallest zero forcing set Z of G such that the graph induced by Z, namely, $G[Z]$ has no isolated vertices. The connected zero forcing number, $Z_C(G)$,

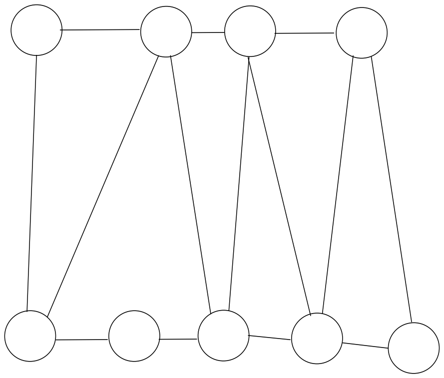

Figure 3.2: A Serpentine Graph

is slightly weaker and is equal to the size of the smallest zero forcing set Z of G such that the graph induced by Z, namely, $G[Z]$ is connected. The following result is presented in [16]. We make use of this result later in this chapter.

Proposition 3.1.5 *If G is a serpentine graph, then $Z(G) = Z_T(G) = Z_C(G) = 2$.*

With the previous definitions and chapters in mind, we can discuss a number of properties about both graph parameters Z and P. First and foremost, they are closely related and can be used to reveal information about one another. For an example of this, consider what we observed in Chapter 2, namely the concept of the z-induced path cover. Now, along these lines, we proved in Chapter 2 that when all minimum path covers of a graph G are z-induced we have $Z(G) = P(G)$. Since the zero forcing number can increase quite rapidly in certain circumstances (such as dense graphs or graphs with a large number of edges), it is more natural to restrict the path cover number to small values when trying to determine membership in the class ZP. In the following section, we explore the zero forcing number and how it relates to certain graphs. Namely, starting with the cycle graph C_n and adding chords in a systematic manner. Without any chords, it is not hard to verify that for $G = C_n$, we have $Z(G) = P(G) = 2$. Beyond this, consideration is given to zero forcing as it relates to the cycle graph with the addition of some interior edges. Various interior chords are examined in different orientations, and in different numbers, ranging from zero chords to MOP. Beyond this, in Section 3, we focus on graphs with path cover number equal to two. As previously noted, in determining membership in ZP, restricting to small values of path cover number is a natural starting point. It is intuitive then, that path graphs have path cover number equal to 1. Thus, considering graphs with path cover number equal to 2 is an obvious next step. In Section 4 of this chapter, we consider graphs with path cover number equal to three and begin an investigation of when such graphs belong to the class ZP. Along these lines are study takes us to the pinwheel graph. It is known that the pinwheel graph fails to be in ZP. In trying to determine the precise impediment concerning membership in ZP for the pinwheel, we examine the concepts of separator triangles and fan-leaf subgraphs. Through our study of the pinwheel and the aforementioned subgraphs, we are able to establish the novel result found in Theorem 3.4.2. Furthermore, there are also a number of related results established throughout the remainder of Chapter 3.

3.2 Cycles with Chords

In the following section, we finally arrive at the research and work accomplished concerning the addition of certain composition of chords starting from an n-cycle. This work was the primary focus early on in the research process and was vital in establishing relations between the zero forcing number and the path cover number of such graphs. In [8] logic circuits are presented as copies of C_3 and C_4 which allowed an examination of various combinations of many cycles on 3 or 4 vertices. We will observe later that these combinations are akin to edge or vertex-sums of cycles, in which the resulting graph is thus a cactus. Now, making use of Proposition 3.1.5, we examined how these parameters are effected if the cycle was changed to a maximal outerplanar graph by adding interior chords, and, in particular, if the orientation of said chords mattered. Ultimately we are able to extend Proposition 3.1.5 and present novel relate and interested results.

We make use of the following definition in Lemma 3.2.2 (see also [18]).

Definition 3.2.1 *A graph G is a graph of two parallel paths if there exists two independent induced paths covers of G that cover all the vertices of G and such that any edges between two paths can be drawn so they do not cross.*

Lemma 3.2.2 *The maximum number, m, of interior chords that can be inserted between the vertices of C_n and preserve the property of being outerplanar is $m = n-3$.*

Proof. We note that in order to connect two previously unconnected vertices, C_n must be on at least 4 vertices. Thus we consider $n = 4$ and $m = 1$ as our base case and proceed by strong induction on n. Now, we consider C_5 C_6 and C_7 with the maximum number of interior chords via the following diagrams: For $n = 5, m = 2$, while for $n = 6, m = 3$. Finally, for $n = 7, m = 4$. As we cannot add another chord while keeping each C_i, $i = 4, \ldots, 7$ outerplanar, we know that our different configurations are maximally outerplanar. We make use of the fact that C_n is a graph of two parallel paths. Thus, we can represent each respective cycle graph in the following ways pictured in Figure 3.3.

We recall that in any maximal outerplanar graph every bounded face (or region in the plane) must be a triangle, otherwise we could add an edge to the graph and preserve the notion of being outerplanar, contradicting maximality. Recall the famous Euler's formula for all planar graphs, namely $n - e + f = 2$ (see [6, 15]), where f is the number of faces in G and e is the number of edges. Observe that if f' is the number of triangular faces then $f' = f - 1$. If we count the edges in the boundary of each face, then we end up counting all of the edges twice. That is $2e = 3f' + n$, the n follows since the unbounded face will have n edges. So now we have: $2e = n + 3(f - 1) = n + 3(2 - n + e - 1) = -2n + 3 + 3e$ or $e = 2n - 3$. Thus it follows that $m = n - 3$. $\blacksquare$

In fact, from geometry, a cycle (C_n) can be seen as an n-sided polygon from which is it is known to be triangulated with $n - 3$ chords and $n - 2$ triangles.

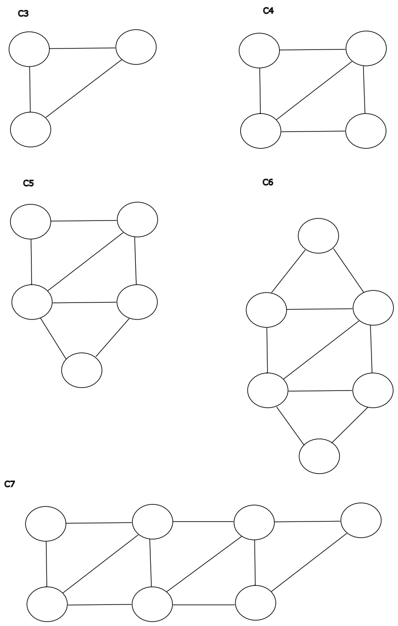

Figure 3.3: Maximally Outerplanar Cycles

Theorem 3.2.3 *If G is an MOP obtained from C_n with $n \geq 4$, by adding some interior chords, then there are exactly two non-adjacent vertices of degree two so long as G is a serpentine graph. If $n = 3$, then all vertices have degree two.*

Proof. It is well known that C_n is a 2-regular graph. We start connecting previously unconnected vertices, up to possible relabelling of the vertices, we can connect vertices

2 and n, then 2 and $n-1$, 3 and $n-1$ etc. We follow this process until we reach our maximum interior number of chords which we have established as $m = n-3$ in the previous theorem. Now, using induction on n, we see each bounded face is a triangle, let the vertices a, b, c form a triangle. Then one of a, b, c must have degree 2, say a. Remove a and now apply induction as the remaining graph is C_{n-1} and MOP. Consequently, the resulting graph H has two non-adjacent vertices, say x and y, of degree two. Observe that we cannot have $\{x, y\} = \{b, c\}$ and it cannot happen that the sets $\{x, y\}, \{b, c\}$ are disjoint, otherwise the graph H is not MOP. Hence exactly one of x or y equals one of b or c. The desired conclusion now follows. ∎

Theorem 3.2.4 *If G is an MOP obtained from C_n with $n \geq 3$, by adding some interior chords so long as G is a serpentine graph, then $Z(G) = 2$.*

Proof. Suppose the vertices of C_n with $n \geq 3$ are given by $V(G) = v_1, \ldots, v_n$. Then $v_{i-1} \sim v_i \sim v_{i+1}$ for $i = 2, \ldots, n-1$. We note here that to connect two previously unconnected vertices, C_n must be on at least 4 vertices.

We consider the following cases and their associated constructions. Clearly if $G = C_n$ has no interior chords, then $Z(G) = P(G) = 2$. If G has one interior chord, we pick an arbitrary $u, v \in V(G)$ such that v is not adjacent to u and join them via an interior chord. We then have the following choices for a possible minimal zero forcing set (abbreviated as ZFS).

Case 1: Both vertices in the zero forcing set are not u or v.

Case 2: One vertex in our zero forcing set is either u or v.

As the vertices in a minimal ZFS must be adjacent to propagate the colouring of C_n, these are the only possibilities for choices.

Proof of Case 1: We select two adjacent vertices that are not part of the interior chord. Each vertex will have only one white neighbour and is free to make forces freely until reaching either u or v. After u or v is coloured, then that respective vertex has two white neighbours and stops making forces until the opposite end of the chord is coloured. Then forces, are free to continue. If by chance u and v are coloured at the same time, the graph colouring proceeds without interruption.

Proof of Case 2: The ZFS contains either u or v. Without loss of generality, pick u. We see that u has two white neighbours so it cannot perform a force. However, the other vertex we have chosen has only one white neighbour and therefore, it is free to perform a force until it colours v at which point it can also continue to force other vertices until the graph is coloured. In either case, we need only select two adjacent vertices to colour G. Thus, $Z(G) = 2$.

Now, if we add more interior chords, we must choose our ZFS more carefully. Consider the case where the interior chords are adjacent. Denote the ends of chord one as v_1 and v_2 and the ends of chord 2 as u_1 and u_2. We see quickly that is we select adjacent vertices u_1 and v_1, that we cannot perform any forces. So, if we select one end of a chord, suppose without loss of generality, we select u, then we choose the vertex which is not part of a chord. That is, the second vertex we select is adjacent to u, but is not v. This leaves us to propagate forces until we reach the end of our first chord which we have coloured. The next time step will colour the vertices of the second chord which leaves the remaining vertices of the graph free to be coloured.

Consider now, two interior chords which are not adjacent. If we choose our ZFS with at least one vertex between the two interior chords, we can perform forces until we reach the ends each respective chord. To clarify this, consider a sequence of four vertices in $V(C_n)$. Denote these vertices u_1, u_2, u_3, u_4. Now, we choose u_2 and u_3 as our ZFS and assume that u_1 and u_4 are connected to two other vertices in $V(C_n)$ which are not adjacent to any other vertices in our sequence. We can now perform two forces; u_2 colours u_1 and u_3 forces u_4. Now, u_1 and u_4 are adjacent to u_2 and u_3 respectively but they are also adjacent to one more vertex each via an interior chord. Thus, u_1 and u_4 both have two white neighbours, and hence we cannot perform any further forces without adding another vertex to our ZFS. Thus, to preserve $Z(G) = 2$ we need to choose our ZFS with at least one vertex outside the ends of the interior chords. That is, a minimium ZFS must contain one of the degree two vertices in G, following Theorem 3.2.3. ▮

So, we see adding a second chord in this manner doesn't change $Z(G)$.

As it turns out, we may add as many chords as possible until C_n is maximally outerplanar. If we select at least one vertex not adjacent to a chord, we are free to propagate forces as shown in the following diagram (see Figure 3.4).

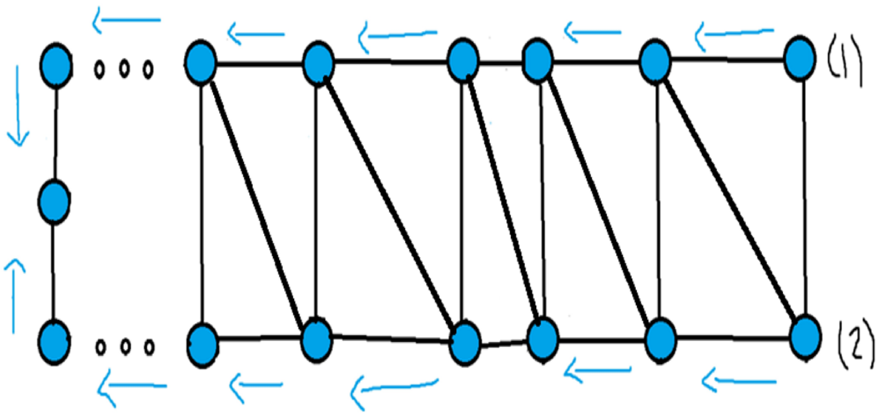

Figure 3.4: Maximally Outerplanar Graph

This is achieved as follows: We select our ZFS where one vertex is not incident with a chord, by construction of a maximally outerplanar graph the second vertex in the ZFS must be part of a chord. Denote the vertex which is not incident with a chord as u and let the other vertex in the ZFS be v_1 with corresponding vertex v_2 being on the other end of the chord. v_1 has u and v_2 as white neighbours and cannot force any vertices. However, as u and v_1 are adjacent, u is free to make forces. Now, by construction u colours v_2 which reduces the number of white neighbours for v_1 to one, but this leaves v_2 with two white neighbours. Thus v_1 now forces it's only white neighbour which reduces the number of white neighbours for v_2 to one. This pattern now continues, alternating colouring the ends of each respective chord, until G is coloured.

The final component of this proof is to show that intersecting chords do not preserve $Z(G) = 2$. There are three cases to consider. Here, let u and v be adjacent vertices of the cycle which are each ends of their respective chords. Figure 3.1 shows intersecting chords and will aid in in the explanation of the following cases.

Case 1: We select the adjacent ends of intersecting chords as the ZFS. We see immediately that each vertex has two white neighbours (the opposite end of the chord and the next adjacent vertex in C_n) and we cannot make any forces.

Case 2: We select the end of one chord, suppose we select u_1, and the adjacent vertex outside the chord. We see that we may perform forces until we reach the other end of the opposite chord, in this case, we have coloured v_2. Now, v_2 is adjacent to v_1 and u_2 and likewise u_1 is adjacent to u_2 and v_1. As both u_1 and v_2 are incident to two white vertices, no further forces can be made and thus $Z(G) \neq 2$.

Case 3: Neither u_1 or v_1 are selected as part of the ZFS and neither are u_1 or v_1. Both vertices in the ZFS are free to make forces until we inevitably reach the end of each chord which produces the same scenario as in Case 2. Thus we cannot make forces.

The conclusion now follows; if our chords intersect, we cannot select a ZFS such that $Z(G) = 2$. Thus, we may add interior chords to C_n up to a maximally outerplanar graph G and still satisfies $Z(C_n) = Z(G) = 2$ so long as there are no intersecting chords. ∎

3.3 On Graphs with Path Cover Number Equal to Two

Recall that a graph G is said to be a member of the class ZP if $Z(H) = P(H)$ for all induced subgraphs H of G.

When the path cover number of a graph is one, then this graph must be a path and thus is a member of the class ZP as all trees are in ZP. We now assume that the path cover number is equal to two. As noted previously, considering graph with path cover number equal to two is a natural next step in examining these graph parameters and the class ZP. Thus, we review a few relevant results and are then able to extend some of these results to a new theorem; Theorem 3.3.3.

Theorem 3.3.1 *[11] Suppose G is a graph. Then $Z(G) = 2$ if and only if G is a graph of two parallel paths.*

Observe that G is a graph of two parallel paths if G is outerplanar and $P(G) = 2$. Hence any serpentine graph is a graph of two parallel paths.

Recall the notions of "terminal" within the contexts of zero forcing chains and path coverings (see also [19] and [5]) from the previous section. A vertex v in a graph G is called *terminal* (either *z-terminal* or *p-terminal*) if it is contained in a minimal zero forcing set or it is the endpoint of a path in a minimal path covering for G. Note that if v is in a minimal zero forcing set then it is the endpoint of a zero forcing chain corresponding that the zero forcing set.

We know recall some facts from [19] and [5, Lemma 2.1]. Suppose G is a graph and v is a vertex in G.

1. $-1 \leq P(G) - P(G-v) \leq 1$,
2. $-1 \leq Z(G) - Z(G-v) \leq 1$,
3. $P(G) = P(G-v) + 1$ if and only if $\{v\}$ is a path in a minimum path covering for G,
4. $Z(G) = Z(G-v) + 1$ if and only if v is in a minimum zero forcing set for G and v performs no forces,
5. If $P(G) = P(G-v) - 1$, then v is not P-terminal,
6. If $Z(G) = Z(G-v) - 1$, then v is not Z-terminal.

We now state an important result concerning the zero forcing number for the vertex sum of two graphs, which can be found in [19]. Recall that if G and H are two graphs, then the graph obtained from G and H by identifying a given vertex v in both G and H is called the *vertex sum of G and H along v* and this operation is denoted by $G \oplus_v H$.

Theorem 3.3.2 *[19] Let G and H be two graphs with identified vertex v. If v is terminal in both G and H, then*

$$Z(G \oplus_v H) = Z(G) + Z(H) - 1.$$

Otherwise

$$Z(G \oplus_v H) = \min\{Z(G) + Z(H - v), Z(G - v) + Z(H)\}.$$

Analogous results also hold when the path cover number is replaced with the zero forcing number above (see [5]).

Now suppose G is a graph that satisfies $Z(G) = P(G) = 2$. Then G is a graph of two parallel paths and we assume that the vertices on this two induced paths are $\{u_1, u_2, \ldots, u_s\}$ and $\{t_1, t_2, \ldots, t_l\}$. To prove that G is a member of the class ZP we use induction on the number of vertices in G, say $|G| = n$. When $n = 3$, the only such connected graph is $G = K_3$ and it is clear that G is in ZP as we know that all cacti are in ZP. Now assume that all graphs G with $Z(G) = P(G) = 2$ on fewer than n vertices are in the class ZP. Now consider the graph G defined above where $s + l = n$. Since $Z(G) = P(G) = 2$, we may assume that H is any proper induced subgraph of G. Hence $H = G \setminus \{u_{i_1}, \ldots, u_{i_x}; t_{j_1}, \ldots, t_{j_y}\}$, where at least one of x are y is at least 1. From the set $\{u_{i_1}, \ldots, u_{i_x}; t_{j_1}, \ldots, t_{j_y}\}$ we select the first u or t (call this vertex w) from the left such that the induced subgraph H' formed from the vertices strictly less than the vertices in this set satisfies $P(H') \leq 2$ (this follows this H' will either be a path or a graph of two parallel paths. In particular, H must be in the form $H = H' \oplus_w H''$, where H' and H'' are both nonempty and $P(H') \leq 2$. Observe that if either H' or H'' are empty then clearly H is a graph on two parallel paths and $Z(H) = P(H)$ as desired. By induction, since H'' is an induced subgraph of a graph of two parallel paths on fewer than n vertices, we have $Z(H'') = P(H'')$. Furthermore, it follows from Theorem 2.3.3 that the "terminal" status of w in H'' is equivalent in both the P and Z concepts.

We now consider two cases depending on $P(H')$.

Case 1: Suppose $P(H') = 1$.

Then H is a path and $Z(H') = P(H') = 1$. If w is terminal in both H' and H'', then by Theorem 3.3.2 we have

$$P(H) = P(H') + P(H'') - 1 = Z(H') + Z(H'') - 1 = Z(H).$$

Otherwise, assume that w is not terminal in at least one of H or H''. Then by Theorem 3.3.2, we have that

$$Z(H) = \min\{Z(H') + Z(H'' - w), Z(H' - w) + Z(H'')\},$$

and, similarly, we have

$$P(H) = \min\{P(H') + P(H'' - w), P(H' - w) + P(H'')\}.$$

Using induction we know that $Z(H'') = P(H'')$ and since H'' is in the class ZP by inheritance, it also follows that $Z(H'' - w) = P(H'' - w)$. Further since H' is a path and in the class ZP we also have $Z(H') = P(H')$ and $Z(H' - w) = P(H' - w)$. Thus in either case above it follows that $Z(H) = P(H)$ as needed.

Case 2: Suppose $P(H') = 2$.

Since H' is outerplanar (as G was outerplanar) it follows that H' is a graph of two parallel paths and hence by induction H' is a member of the class ZP. Furthermore, $H' - w$ is an induced subgraph of H' and hence $H' - w$ is a member of the class ZP. Applying similar arguments as in Case 1, it follows that $Z(H) = P(H)$, which completes the proof. ∎

The above analysis establishes that following result.

Theorem 3.3.3 *Suppose G is a graph with path cover number equal to two. Then G is a member of the class ZP if and only if G is a graph of two parallel paths (that is, $Z(G) = 2$).*

3.4 Separator Triangles, Pinwheels and Graphs with Path Cover Number Three

In an effort to classify the graphs in the class ZP, we seek to add to the list of known forbidden subgraphs for membership in ZP. To this end, we examine the pinwheel graph which is pictured below in Figure 3.5.

Upon initial examination, we did suspect that the pinwheel would indeed be in ZP. However, after subsequent attempts to find z-induced path covers, we came to the known conclusion that $Z(G) > P(G)$, see [3, Example 2.11] for the pinwheel graph, denoted by G for convenience. If we consider $P(G)$, the graph may be covered in 3 vertex disjoint paths. These are from vertices 12 to 1, then 2 to 9 and finally 8 to 10. Now we consider $Z(G)$. If the pinwheel were in ZP our path cover would be

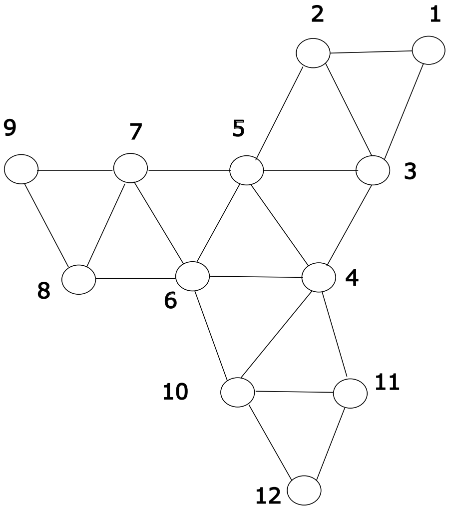

Figure 3.5: The Pinwheel Graph

z-induced. So, choose two of the ends of any of the paths. Without loss of generality, we choose vertices 10 and 12 as our ZFS and start performing forces. Vertices $11, 4$ and 6 are coloured. However, at this point both vertices 4 and 6 have at least 2 white neighbours. Now, we add vertex 2 to our ZFS as it is the end of one of our path covers. However, this does not allow us to colour any more vertices. Notice that vertex 2 has 3 white neighbours, vertices $1, 3$ and 5. Hence we come to the known result; the Pinwheel is not in ZP, but we can say more. At the time-step where we have just coloured vertex 6 we cannot colour the graph by adding one more vertex to our ZFS, regardless of our choice of vertex. Thus we add a fourth vertex to our

ZFS, vertex 1 will do, and we are free to colour the rest of the graph. It is known that $Z(G) > P(G)$, in particular, $Z = 4$ and $P = 3$ (see [3, Example 2.11]). Now, as was noted previously, the separator triangle was the subgraph which was thought to violate the relationship that $Z(G) = P(G)$.

Naturally, we wonder if there exists another subgraph or likewise structure that will allow us to ensure membership in ZP even if there is a separator triangle in G. As it turns out, such a thing does exist. The following can once again be found in [16]. We define a *serpentine leaf* as the set of triangles in a graph G that form the only path in H connecting a leaf of H with a vertex u of H such that $deg(u) > 2$. The number of serpentine leafs with fan structure from a vertex of a separator triangle, called a *fan-leaf*, is denoted by h_f. In the following diagram (see Figure 3.6), we see that the so called fan structure allows us to bypass the the separator triangle.

Thus, we can conclude that the problem with the pinwheel is not that it contains a separator triangle. Rather, it is the lack of a fan-leaf structure which precludes the pinwheel graph from being in ZP.

Upon examination of the pinwheel and the various choices of vertices for the ZFS another point of interest arises in the form of the following.

Lemma 3.4.1 *Denote the pinwheel graph depicted in Figure 3.8 as* G. *Then,* $G - v \in ZP$, *for any vertex* v.

Proof. Consulting the vertex labelling in Figure 3.8, we initially consider three cases,

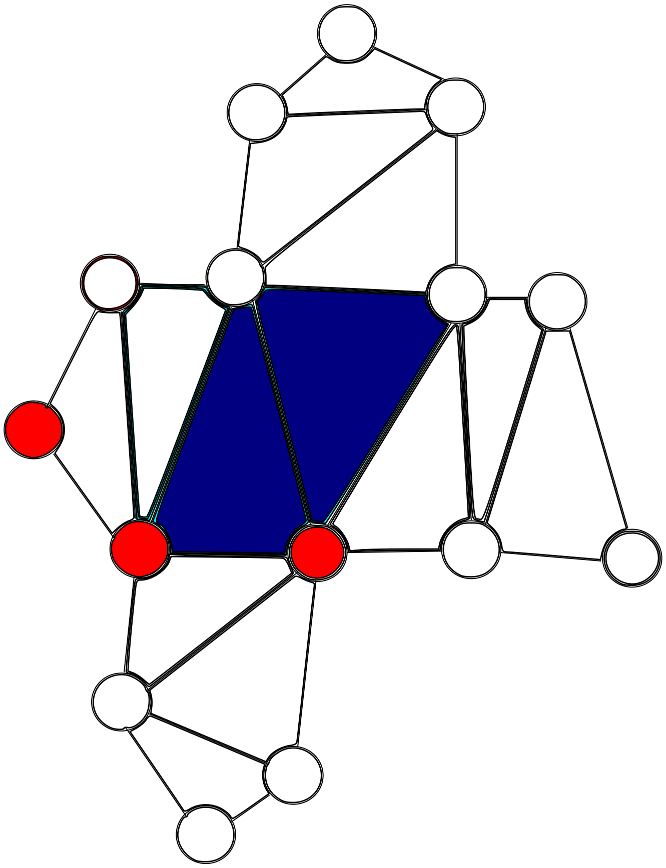

Figure 3.6: A maximally outerplanar graph with highlighted fanleaf structure with $h_f = 1$.

namely that we remove vertex 1, or vertex 2, or vertex 5. Observe that removing any other single vertex i from the pinwheel for $i \neq 1, 2, 5$ results in a graph obtained in one of the four cases above by considering the basic symmetries of the pinwheel graph.

First consider removing vertex 1. Then the resulting graph is an MOP with $P(G) = 3$ and $h_f \geq 1$. Then by Proposition 2 in [16] we have $Z(G) = P(G) = 3$.

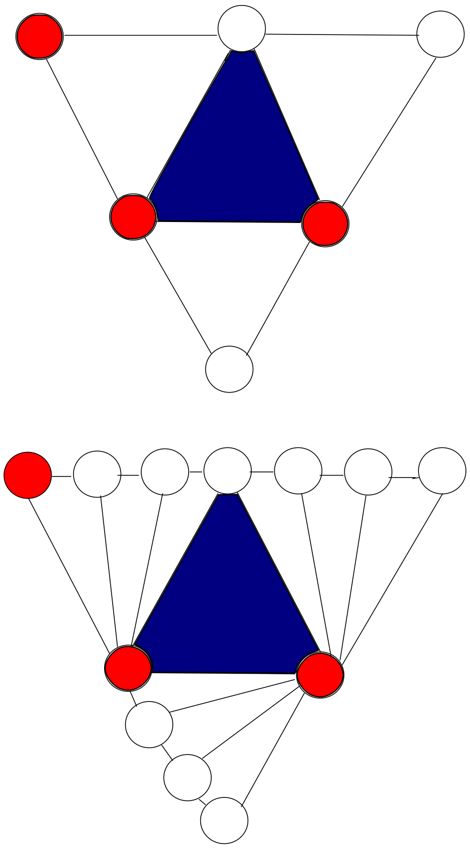

Figure 3.7: Fanleafs ($h_f = 3$)

Further if vertices 2 or 3 are then removed, the same result applies. If other vertices are removed, then the resulting graph is a vertex sum of two graphs: one is a (possibly induced subgraph of a) graph of two parallel paths and the others is a tree. From the work in Chapter 4 (a forward logical reference) this graph is in ZP. Applying similar arguments, we observe that removing vertex 2, results in a graph that is a vertex sum (at vertex 4) of a graph of two parallel paths and a path, and again this graph is in ZP following the work in Chapter 4. Finally, removing vertex 5, we are

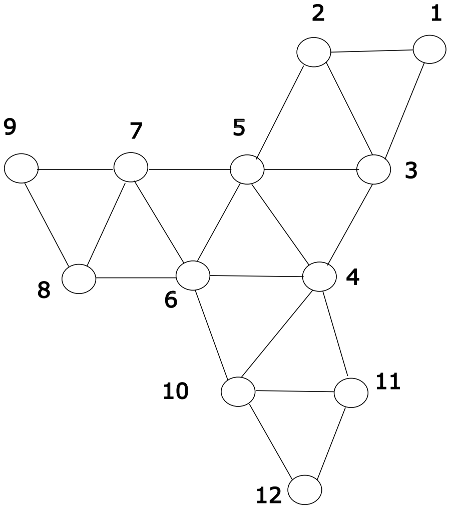

Figure 3.8: The Pinwheel Graph

left with a graph that is vertex sum (at vertex 4) of an induced subgraph of a graph of two parallel paths and a unicyclic graph (which is a cactus), both of which belong to the class ZP, and hence the subgraph in question belongs to ZP.

Now, after considering the above lemma, what would our forbidden subgraph in the pinwheel be? Consider the triangle composed of vertices $4, 5, 6$. Notice that none of the edges in this triangle are adjacent to the infinite face. Initially, this seemed to be the forbidden subgraph that would preclude the pinwheel from being in ZP.

However, we have just shown that $G - v$ is indeed in ZP. We have left the separator triangle intact with no edges adjacent to the infinite face, this cannot be the a forbidden subgraph for ZP. As we now know that we may bypass a separator triangle with a fan-leaf it is the case that for the pinwheel, there are simply too many vertices to colour such that $Z(G) = 3$.

We note here that the pinwheel graph is an MOP graph with path cover number 3, $h_f = 0$, and does not belong to the class ZP. However, for such MOP graphs with $h_f > 0$ we can say much more in the affirmative.

Theorem 3.4.2 *Suppose G is an MOP with $P(G) = 3$. Then G is in ZP if and only if h_f is at least one.*

Proof. For the forward implication, assume that G is in ZP. Then $Z(G) = 3$. Thus using [16, Proposition 2] it follows that $h_f \geq 1$. On the other hand, suppose G is an MOP with $P(G) = 3$ and $h_f \geq 1$. Then by Proposition 2 in [16] we have $Z(G) = P(G) = 3$. Let H be an induced subgraph of G. There are two cases to consider.

Case 1: $t(H) = 0$. In this case it follows that H is either a graph on two parallel paths or is an induced subgraph of a graph on two parallel paths. In either case H belongs to ZP by Theorem 3.3.3. Hence $Z(H) = P(H)$, as needed.

Case 2: $t(H) \geq 1$. Now, if we know H contains a fan leaf, then $Z(H) = 3$ by Proposition 2 in [16]. So if $P(H) = 3$, we are done. Otherwise $P(H)$ is at most 2. Since the path is the only graph with path cover equal to one, we may assume that $P(H) = 2$. In this case H is a graph on two parallel paths and is in ZP, which is a contradiction. Now assume that H has no fan leaves. In this case we observe that the pinwheel is the smallest graph with $t = 1$ and $h_f = 0$, since we have assume that the number of fan leaves of G is at least one it is not possible to realize the pinwheel as a subgraph of G. This completes the proof. $\blacksquare$

Chapter 4

Graph Operations and the Class ZP

4.1 Motivation and Background

In this chapter, we study various graph operations associated with the class ZP. As in previous chapters, we begin this discussion with some relevant definitions and figures to illustrate these operations. A quick note, the examples provided for such graph operations that preserve ZP are not intended to act serve as rigorous proofs, but are only to provide context to each related statement.

We now define five key graph operations, the first three of which are rather standard and can be found in most introductory graph theory textbooks. The vertex- and edge-sums of graphs are less-known but have been studied before (see [5], for example).

Definition 4.1.1 *Suppose we are given two graphs* $G_1 = (V(G_1), E(G_1))$ *and* $G_2 =$

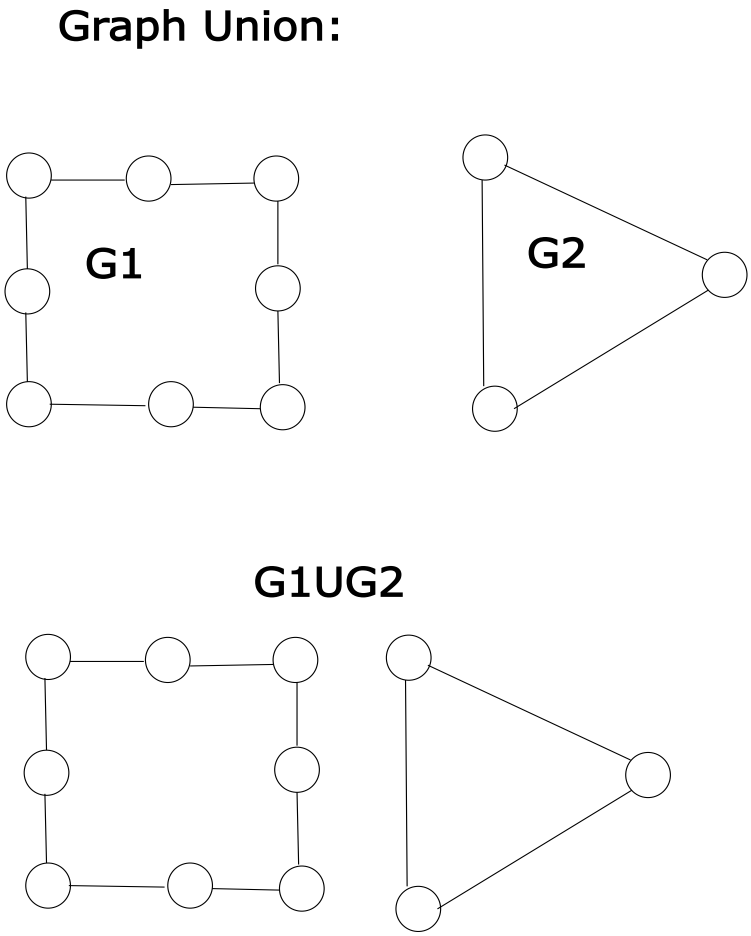

Figure 4.1: Graph union

$(V(G_2), E(G_2))$, where $V(G_1)$ and $V(G_2)$ are disjoint. Then their union $G_1 \bigcup G_2$ is the graph with the vertex set $V(G_1) \bigcup V(G_2)$ and edge set $E(G_1) \bigcup E(G_2)$.

Definition 4.1.2 *The Cartesian product (also known as the box product) of a graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph with vertex set $V(G) \times V(H)$*

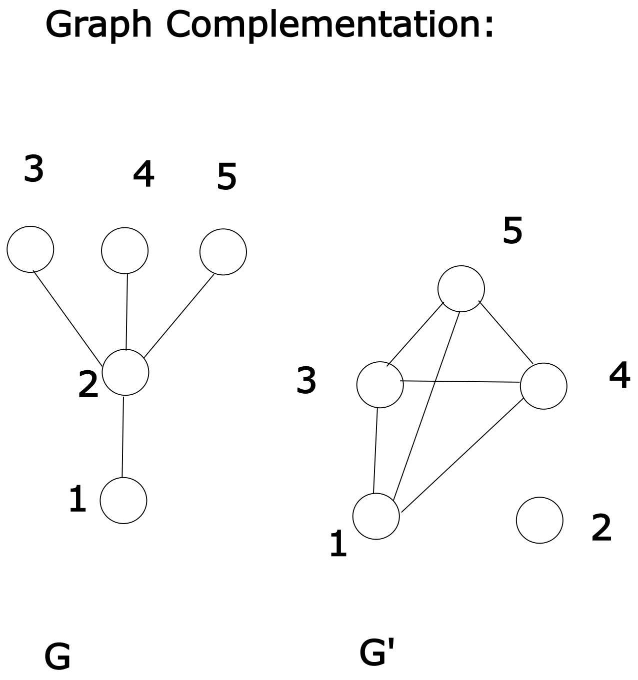

Figure 4.2: Graph complementation

such that (u, v) *is adjacent to* (u', v') *if and only if* $u = u'$ *and* $uv' \in E(H)$, *or* $v = v'$ *and* $uu' \in E(G)$. *The Cartesian product of* G *and* H *is denoted by* $G\Box H$.

Definition 4.1.3 *Consider a graph denoted by* $G = (V(G), E(G))$. *Then the complement of a graph* G, *denoted* $\overline{G}$, *is the graph with vertex set* $V(G)$ *in which two vertices are adjacent if and only if the are not adjacent in* G.

Definition 4.1.4 *Vertex Sum: Let* $G_1 \ldots G_h$ *be disjoint graphs. For each* i, *we select a vertex* $v_i \in V(G_i)$ *and join all the* G_i*'s by identifying all the* v_i*'s as a unique vertex*

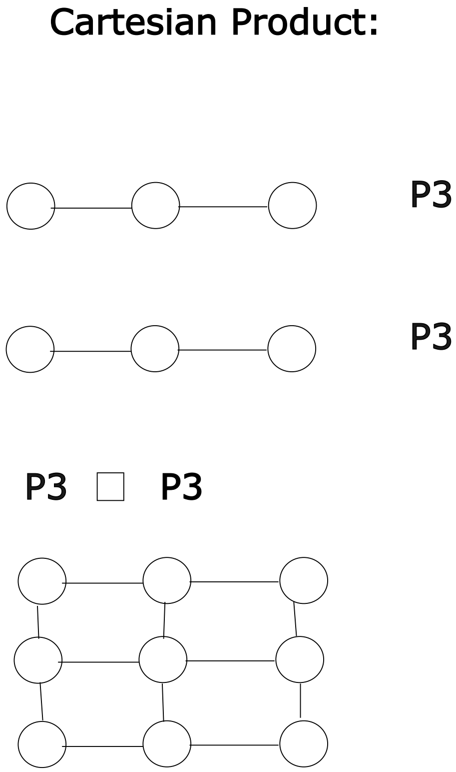

Figure 4.3: Graph Cartesian product

v. The resulting graph is called the vertex-sum at v of the graphs $G_1 \ldots G_h$.

Definition 4.1.5 *The edge-sum of two graphs is defined as follows: let G_1 and G_2 be disjoint graphs, and let v_1 and v_2 be vertices of G_1 and G_2 respectively. If we connect G_1 and G_2 by adding the edge $e = \{v_1, v_2\}$, the resulting graph G is called the edge-sum of G_1 and G_2 along the edge e.*

It is interesting to note that some of the graph operations preserve membership in

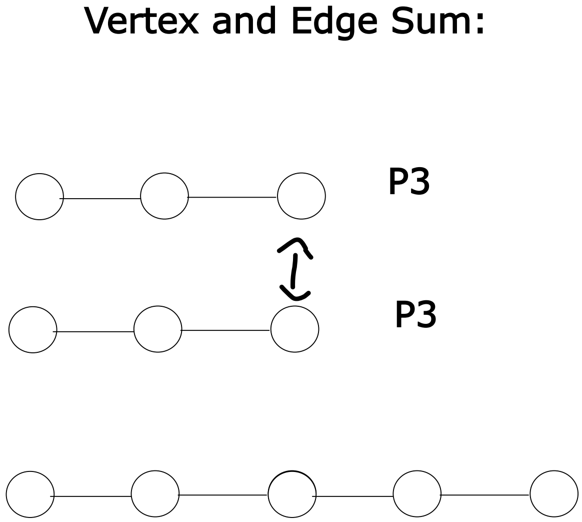

Figure 4.4: Vertex sum

ZP in general while others do not. In some of the associated figures, we will produce examples of the graph operations which do not preserve membership in ZP. Later in this section, in relation to the vertex- and edge-sums of graphs, we establish that these operations are closed within the class ZP. First, we examine the graph union operation as illustrated in Figure 4.1. It is not hard to see that in this particular example, membership in ZP is preserved. Indeed, we have two cycles, which we have established are in ZP (see Chapter 3). More generally, since both the zero forcing number and the path cover number sum over connected components of a disconnected graph, the following implication is straightforward. Suppose G_1 and G_2

are two connected graphs that belong to ZP and let $G = G_1 \cup G_2$. Further, suppose H is any induced subgraph of G. Then $H = H_1 \cup H_2$, where H_i is an induced subgraph of G_i, for $i = 1, 2$. Then we have

$$P(H) = P(H_1) + P(H_2) = Z(H_1) + Z(H_2) = Z(H),$$

where the second equality follows from the assumption that both G_1 and G_2 belong to ZP. From this, it is sufficient to study the case of connected graphs concerning membership in the class ZP.

Now, referring to Figure 4.2, we examine graph complementation. For the graph G,, since G is a tree with know G is in ZP, and we have $Z(G) = P(G) = 3$. However, in G',, we observe that $Z(G) = 4 \neq 3 = P(G)$. Thus, the operation graph complementation does not necessarily preserve membership in ZP. Similarly, referring to Figure 4.3, we note that the Cartesian or box product also does not preserve membership in ZP. Upon examination of Figure 4.3, it is clear to see that each copy of the graph P_3 is in ZP and has $Z(G) = P(G) = 1$. However, in $P_3 \Box P_3$ to compute $Z(G)$, we must select an entire row or column which does not contain the middle vertex. We are then free to colour the entire graph. While computing the path cover, we start with any vertex on the outside of the graph, so, without loss of generality, we select the vertex on the bottom right. We then select a path from this vertex, along the bottom row, then up the left most column, and finally across the top row. The remaining two vertices are covered by a path which yields $P(G) = 2$.

Hence, the Cartesian product need not preserve ZP.

Finally, we come to the vertex-sum and edge-sum of graphs. For the vertex-sum, consider Figure 4.4. Once again, using two copies of P_3 as G and H respectively, we have G and H are in ZP, and that $Z(G) = P(G) = Z(H) = P(H) = 1$. In this case we identify one vertex on the end of each graph, and in taking the vertex sum, we produce the graph P_5, which also clearly in ZP. Now, if we consider the edge-sum using P_3 as G and H once again, we can produce a similar result. Identifying one vertex on each end as indicated in Figure 4.4, we join the identified vertices via an edge. Thus, we create the graph P_6 as our new graph, which is once again in ZP. In the next section we will prove that the vertex sum and edge sum are two graph operations that are closed within the class ZP.

4.2 Certain Graph Operations on ZP

In this section we prove some key results concerning a mechanism to generate graphs in the class ZP.

Along the way we establish two novel results for this class and extend some known results from literature that represent the current state of the art on the class ZP.

Recall that if G and H are two graphs, then the graph obtained from G and H by identifying a given vertex v in both G and H is called the *vertex sum of G and H along v* and this operation is denoted by $G \oplus_v H$.

Theorem 4.2.1 *Let G and H be two nonempty graphs with identified vertex v. Suppose both G and H belong to the class ZP. Then $G \oplus_v H$ belongs to ZP.*

Proof. The proof uses induction on the parameter $|G| + |H|$. For the base case we may assume $|G| + |H| = 2$. (May be for good measure you could work out the cases when $|G| + |H|$ is at most 4). In this case one of $|G|$ and $|H|$ is 2 and the other is equal to 1. Hence $G \oplus_v H$ is K_2 and is in ZP. So assume the result holds for all such graphs G and H with $|G| + |H| < N$. Now assume that G and H are two graphs belonging to ZP such that $|G| + |H| = N$.

Let K' be a proper induced subgraph of $K = G \oplus_v H$. Assume that $K' = K \setminus \{v_1, v_2, \ldots, v_s\}$, where $s \geq 1$. There are two cases to consider.

Case 1: $v \in \{v_1, v_2, \ldots, v_s\}$.

In this case we may assume that $v = v_l$ and that $\{v_1, \ldots, v_l\}$ is in G and $\{v_l, \ldots, v_s\}$ is in H. Hence K' is disconnected and $K' = G' \cup H'$, where $G' = G \setminus \{v_1, \ldots, v_l\}$ and $H' = H \setminus \{v_l, \ldots, v_s\}$. In this case we have $Z(K') = Z(G') + Z(H')$ and $P(K') = P(G') + P(H')$. Since G and H are both in ZP it follows that $Z(G') = P(G')$ and $Z(H') = P(H')$. Hence $Z(K') = P(K')$.

Case 2: $v \notin \{v_1, v_2, \ldots, v_s\}$.

As in the previous case, assume that $\{v_1, \ldots, v_l\}$ is in G and $\{v_{l+1}, \ldots, v_s\}$ is in H. Now in this case we have $K' = G' \oplus_v H'$, where $G' = G \setminus \{v_1, \ldots, v_l\}$ and $H' = H \setminus \{v_{l+1}, \ldots, v_s\}$. Hence K' is the vertex sum of two graphs G' and H' with both G' and H' in ZP and $|G'| + |H'| = N - s < N$, as $s \geq 1$. So, by the induction hypothesis we have $Z(K]) = P(K')$.

Thus in either case we have $Z(K') = P(K')$.

All that remains to be proved is that $Z(K) = P(K)$.

Since G and H both belong to the class ZP, we know from Theorem 2.3.3 that the terminality of any vertex is equivalent for zero forcing chains and minimum path coverings. Thus, for K, we consider the following cases:

Case 1: Assume v is terminal in both G and H. Then we have using Theorem 3.3.2 that

$$Z(K) = Z(G) + Z(H) - 1 = P(G) + P(H) - 1 = P(K).$$

Case 2: Assume v is not terminal in either G or H. Then

$$Z(K) = \min\{Z(G) + Z(H - v), Z(G - v) + Z(H)\},$$

and

$$P(K) = \min\{P(G) + P(H - v), P(G - v) + P(H)\}.$$

Since both G and H are in ZP, we have $Z(G) = P(G)$, $Z(H - v) = P(H - v)$, $P(G-v) = Z(G-v)$, and $Z(H) = P(H)$. Hence it follows again that $Z(K) = P(K)$.

∎

We now list a number of interesting and useful consequences of our main result (Theorem 4.2.1).

Corollary 4.2.2 *Suppose* $G = (V, E)$ *is a graph in* ZP*. Then the graph* H *obtained from* G *with vertex set* $V \cup \{x\}$ *and edge set* $E \cup \{v, x\}$*, where* v *is any vertex in* G *is in* ZP*.*

Proof. This follows since H can be viewed as a vertex sum of G and K_2. Both of which are in ZP. ∎

We now consider another graph operation known as the edge sum of two graphs. Suppose G_1 and G_2 be two disjoint graphs, and let v_1 and v_2 be vertices of G_1 and G_2, respectively. If we connect G_1 and G_2 by adding the edge $e = \{v_1, v_2\}$, the resulting graph is called the *edge-sum* of G_1 and G_2, and is denoted by $G \oplus_e G_2$.

Corollary 4.2.3 *Suppose* G_1 *and* G_2 *are two graphs in* ZP*. For any vertex* v_1 *in* G_1 *and any vertex* v_2 *in* G_2*, we have that Then* $G_1 \oplus_e G_2$ *is in* ZP*, where* $e = \{v_1, v_2\}$*.*

Proof. Using the hypotheses, set $H = G_1 \oplus_e G_2$. Now observe that H can be viewed as the vertex sum of G_1 and the graph G_2' where G_2' is obtained from G_2 by appending a pendant vertex v_1. By Corollary 4.2.2 it follows that G_2' is in ZP. Finally, we have $H = G_1 \oplus_{v_1} G_2'$, and hence by Theorem 4.2.1 we have that H is in ZP. ∎

Using the work above, we can re-establish the strongest known result concerning the class ZP (see also [14, 21]). Recall that a graph G is called a cactus if every two distinct cycles in G are edge-disjoint.

Corollary 4.2.4 *Suppose* G *is cactus graph. Then* G *in* ZP*.*

Proof. It is straightforward to observe that any cactus graph can be constructed as a vertex sum of cycles and trees, both of which are in ZP. Hence, by Theorem 4.2.1, any cactus is in ZP. ∎

Chapter 5

Conclusions and Future Considerations

In the preceding work of this thesis, we explored various graph parameters, primarily the zero forcing and path cover numbers. The relation between these graph parameters was studied, and this was done primarily through examination of the class ZP, in which the zero forcing and path cover numbers are equal over all induced subgraphs. The purpose of the various research and examination that we conducted was to extend existing concepts and research by contributing novel results. Further, the main open problem which was explored, was to fully classify the class of graphs ZP. While the largest novel result of this thesis does help in this regard, there is certainly more work that can be accomplished on this class of graphs.

In Chapter 2, previously known relevant facts and history of the central graph parameters were stated as they pertained to the problems that were being explored in this thesis. Of particular note in this chapter, was defining ZP, established work by Row in [21] and [20] concerning cacti, and notions of terminality from both [19] and [5]. This ensured that all general housekeeping was complete before proceeding with Chapter 3, in which our novel research would start to be presented.

Chapter 3 focused on cycle graphs with interior chords in different configurations. In conjunction with the zero forcing number on these cycles, we were able to build upon known results on serpentine graphs while exploring how the zero forcing number behaved on maximally outerplanar graphs. During this examination, the class of graphs with path cover number equal to two naturally came into play. Results from [11] and [19] were presented, and the intuitive relation between graphs of two parallel paths started to emerge, with Theorem 3.3.3 confirming this notion. That is, if G is a graph of two parallel paths, then it is in ZP. Finally, in Section 3.4 we explore the pinwheel graph as well as important subgraphs including separator triangles and fanleafs. This was done in an effort to see if there was a particular subgraph structure that disqualified the pinwheel from membership in ZP, which in turn allowed us to gain insight into some graphs with path cover number three that may lie in the class ZP.

In Chapter 4, various graph operations are defined, but only vertex and edge-sums are examined closely as they pertain to ZP. Though many standard graph operations need not preserve membership to ZP in general, the vertex and edge-sum operation does indeed preserve membership as is confirmed in Theorem 4.2.1. This result allows us to streamline the result originally published by Row [20], that is, that cacti are in ZP. Corollary 4.2.4 also shows that, as cacti can be constructed as vertex-sums of cacti and trees, and thus are in ZP, but the proof technique is rather different and perhaps more interesting.

As we reflect on the work one in this thesis, there are still some interesting open problems which warrant further examination. First and foremost, as mentioned, is to completely classify the class ZP. In accordance with that, a complete list of forbidden subgraphs is also sought for ZP, and such an investigation will require a more careful examination of the core components of graph that force minimum path covers to be z-induced. Further, more exploration into graphs with path cover number equal to three is needed moving forward. Theorem 3.4.2 does start to touch on this, but certainly more can be said on this particular subset of graphs. Some of the first papers that we reviewed in the research process were in the areas of physics and computer science. In terms of physics, there was a given relation between zero forcing and quantum systems shown in [7]. This struck me as an interesting application for the zero forcing number. As quantum physics can be very abstract at times, having a potential visual representation for quantum systems could potentially be quite helpful. There is a historical precedent for visual representations of processes in the quantum state, as certainly, any physicist would recognize a Feynman diagram. One wonders if there may be a similar representation that exists using zero forcing.

Beyond this, zero forcing has been shown to be a promising concept in other areas of science. In [8] logic circuits were examined by Burgarth et al. As zero forcing has shown to be widely applicable in multiple areas of science, it is not hard to imagine that further examination of the zero forcing number may produce further applicable parameters across the sciences.

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